Copositivity of a class of fourth order symmetric tensors *

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Abstract

In this paper, we mainly dicuss the analytic conditions of copositivity of a class of 4th order 3-dimensional symmetric tensors. For a 4th order 3-dimensional symmetric tensor with its entries 1 or -1, an analytic necessary and sufficient condition is given for its strict copositivity with the help of the properties of strictly semi-positive tensors. And by means of usual maxi-min theory, a necessary and sufficient condition is established for copositivity of such a tensor also. Applying these conclusions to a general 4th order 3-dimensional symmetric tensor, the analytic conditions are successfully obtained for verifying the (strict) copositivity, and these conditions can be very easily parsed and validated.

Keywords: Copositivity, Fourth order tensors, Homogeneous polynomial.

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1 Introduction

One of the most direct applications of 4th order copositive tensors is to verify the vacuum stability of the Higgs scalar potential model

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[1–6]. In the graph theory, the mth order copositive tensors may be directly applied to estimate the bounds on the independent number of m-uniform hypergraph [3, 7–9]. The concept of copositive tensors was introduced by Qi [10] in 2013, which is usually applied to a symmetric tensor or, more precisely, to its associated Homogeneous polynomial of degree m.

Definition 1.1. *Let $\mathcal{T} = (t_{i_1i_2\cdots i_m})$ be an mth order n dimensional symmetric tensor. $\mathcal{T}$ is called*

(i) **positive semi-definite** *([11]) if m is an even number and in the Euclidean space $\mathbb{R}^n$, its associated Homogeneous polynomial*

$$\mathcal{T}x^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} t_{i_1i_2\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m} \geq 0;$$

(ii) **positive definite** *([11]) if m is an even number and $\mathcal{T}x^m > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$;*

(iii) **copositive** *([10]) if $\mathcal{T}x^m \geq 0$ on the nonnegative orthant $\mathbb{R}^n_+$;*

(iv) **strictly copositive** *([10]) if $\mathcal{T}x^m > 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$.*

Clearly, the positive semi-definite tensors must be copositive, and a copositive tensor coincides with a copositive matrix if $m = 2$. The concept of copositive matrices was introduced by Motzkin [12] in 1952. Baston [13] gave an analytic way of judging copositivity of a $n \times n$ matrix in 1969.

Theorem 1.1. *(Baston [13]) Let $M = (m_{ij})$ be a symmetric matrix with $|m_{ij}| = m_{ii} = 1$ for all $i, j \in \{1, 2, \cdots, n\}$. Then the matrix M is copositive if and only if there is no triple (r, s, t) such that*

$$m_{rs} = m_{rt} = m_{st} = -1.$$

Simpson-Spector [14] and Hadeler [15] and Nadler [16] and Chang-Sederberg [17] and Andersson-Chang-Elfving [18] respectively showed the (strict) copositive conditions of 2×2 and 3×3 matrices using different methods of argumentation.

Theorem 1.2. *Let $M = (m_{ij})$ be a symmetric matrix. Then a 2×2 matrix M is (strictly) copositive if and only if*

$$m_{11} \geq 0\ (> 0), m_{22} \geq 0\ (> 0), m_{12} + \sqrt{m_{11}m_{22}} \geq 0\ (> 0);$$

a 3×3 matrix M is (strictly) copositive if and only if for all $i \in \{1, 2, 3\}$,

$$\begin{aligned} m_{ii} \geq 0\ (> 0), \alpha = m_{12} + \sqrt{m_{11}m_{22}} \geq 0\ (> 0),\\ \beta = m_{13} + \sqrt{m_{11}m_{33}} \geq 0\ (> 0), \gamma = m_{23} + \sqrt{m_{33}m_{22}} \geq 0\ (> 0),\\ m_{12}\sqrt{m_{33}} + m_{13}\sqrt{m_{22}} + m_{23}\sqrt{m_{11}} + \sqrt{m_{11}m_{22}m_{33}} + \sqrt{2\alpha\beta\gamma} \geq 0\ (> 0). \end{aligned}$$

Schmidt-Heβ [19] provided the nonnegative conditions of a cubic and univariate polynomial with real coefficients in non-negative real number $\mathbb{R}_+$, and Qi-Song-Zhang [20] recently gave the positivity of such a cubic polynomial, which actually gave a the (strict) copositivity of 3rd order 2-dimensional symmetric tensor (see Liu-Song [21] for more details also). Qi-Song-Zhang [20] also gave the nonnegativity and positivity of a quartic and univariate polynomial in $\mathbb{R}$, which means the positive (semi-)definitiveness of 4th order 2-dimensional tensor. Ulrich-Watson [22] and Qi-Song-Zhang [20] presented the analytic conditions of the nonnegativity of a quartic and univariate polynomial in $\mathbb{R}_+$. This actually yielded the copositivity of 4th order 2-dimensional symmetric tensor [4].

Theorem 1.3. *A 3rd order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijk})$ is (strictly) copositive if and only if $t_{111} \geq 0$ (> 0), $t_{222} \geq 0$ (> 0), either $t_{112} \geq 0,\ t_{122} \geq 0$ or*

$$4t_{111}t_{122}^3 + 4t_{112}^3t_{222} + t_{111}^2t_{222}^2 - 6t_{111}t_{112}t_{122}t_{222} - 3t_{112}^2t_{122}^2 \geq 0 \ (> 0).$$

A 4th order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ with $t_{1111} > 0$ and $t_{2222} > 0$ is copositive if and only if

$$\begin{cases} \Delta \leq 0, t_{1222}\sqrt{t_{1111}} + t_{1112}\sqrt{t_{2222}} > 0; \\ t_{1222} \geq 0, t_{1112} \geq 0, 3t_{1122} + \sqrt{t_{1111}t_{2222}} \geq 0; \\ \Delta \geq 0, \\ |t_{1112}\sqrt{t_{2222}} - t_{1222}\sqrt{t_{1111}}| \leq \sqrt{6t_{1111}t_{1122}t_{2222} + 2t_{1111}t_{2222}\sqrt{t_{1111}t_{2222}}} \\ (i) \ -\sqrt{t_{1111}t_{2222}} \leq 3t_{1122} \leq 3\sqrt{t_{1111}t_{2222}}; \\ (ii) \ t_{1122} > \sqrt{t_{1111}t_{2222}} \text{ and} \\ t_{1112}\sqrt{t_{2222}} + t_{1222}\sqrt{t_{1111}} \geq -\sqrt{6t_{1111}t_{1122}t_{2222} - 2t_{1111}t_{2222}\sqrt{t_{1111}t_{2222}}}, \end{cases}$$

where $\Delta = 4 \times 12^3(t_{1111}t_{2222} - 4t_{1112}t_{1222} + 3t_{1122}^2)^3$
$-72^2 \times 6^2(t_{1111}t_{1122}t_{2222} + 2t_{1112}t_{1122}t_{1222} - t_{1122}^3 - t_{1112}^2t_{2222} - t_{1111}t_{1222}^2)^2$.

For a special 4th order 3-dimensional tensor given by the particle physical model, Qi-Song-Zhang [23] presented a necessary and sufficient condition of copositivity, and Song-Li [4] provided an analytic condition of its copositivity. However, an analytic necessary and sufficient conditions has not been found for the copositivity of a general 3-dimensional higher order tensor ($m > 2$). Even for a general 2-dimensional higher order tensor ($m > 3$), people still has not found the analytic conditions of strict copositivity untill now.

For checking the copositivity of symmetric tensors, various numerical algorithms have been employed. For example, Chen-Huang-Qi [3, 24] gave the detection algorithms based on simplicial partition; Li-Zhang-Huang-Qi [25] used an SDP relaxation algorithm; Nie-Yang-Zhang [7] devised a complete semi-definite algorithms. Taking advantage of the properties of a tensor itself, the copositivity can be

described qualitatively. Song-Qi [26] showed a necessary and sufficient condition of copositivity by means of the principal sub-tensors; Song-Qi [27] applied the sign of its Pareto H-eigenvalue (Z-eigenvalue) to test the copositivity. Song-Qi [28] proved the equivalence of (strict) copositivity and (strict) semi-positivity of a symmetric tensor.

Theorem 1.4. *(Song-Qi [28]) Let $\mathcal{T} = (t_{i_1 i_2 \cdots i_m})$ be a symmetric tensor. Then $\mathcal{T}$ is (strictly) copositive if and only if $\mathcal{T}$ is (strictly) semi-positive, i.e., for each $x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n_+ \setminus \{0\}$, there exists $k \in \{1, 2, \cdots, n\}$ such that*

$$x_k > 0 \text{ and } (\mathcal{T}x^{m-1})_k = \sum_{i_2 \cdots i_m = 1}^{n} t_{k i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \geq 0 \ (> 0).$$

For more details of the copositivity of a higher order tensor and a matrix, see [8, 29–37]. Since there are strong connection between the semi-positivity of a tensor and the tensor complementarity problems [26, 38–40], so the copositivity of a symmetric tensor may be verified by solving the corresponding tensor complementarity problems. For more details about the tensor complementarity problems and its applications, see Refs. [41–59].

Motivation to checking the copositivity of a higher order tensor, we mainly dicuss analytic necessary and sufficient conditions of copositivity of a class of 4th order 3-dimensional symmetric tensors in this paper. With the help of Theorem 1.4, we first promote Theorem 1.1 to ones of 4th order 3-dimensional symmetric tensors, which gives an analytic necessary and sufficient condition of strict copositivity of such a class of tensors (Theorem 3.4). Secondly, we present an analytic necessary and sufficient condition of copositivity of such a class of tensors (Theorem 3.8). Finally, applying Theorems 3.4 and 3.8, the analytic sufficient conditions (Theorems 3.9, 3.10 and 3.11) are successfully proved for (strict) copositivity of a general 4th order 3-dimensional symmetric tensor also.

2 Copositivity of 4th order 2-dimensional symmetric tensors

Let $T = (t_{i_1 \cdots i_m})(i_j = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$ be a mth-order n-dimensional symmetric tensor. Then for $x = (x_1 . x_2, \cdots, x_n)^\top \in \mathbb{R}^n$, we write

$$Tx^m = x^\top (Tx^{m-1}) = \sum_{i_1 \cdots i_m = 1}^{n} t_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m},$$

and $Tx^{m-1} = (y_1, y_2, \cdots, y_n)^\top$ is a vector with its components

$$y_k = (Tx^{m-1})_k = \sum_{i_2 \cdots i_m = 1}^{n} t_{ki_2\cdots i_m} x_{i_2} \cdots x_{i_m},\ k = 1, 2, \cdots, n.$$

Let $f(x_1, x_2)$ be a quartic homogeneous real polynomial about two variables x_1, x_2,

$$f(x_1, x_2) = x_1^4 + 4ax_1^3x_2 + 6bx_1^2x_2^2 + 4cx_1x_2^3 + x_2^4. \tag{1}$$

Then it gives a 4th-order 2-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ with its entiries,

$$t_{1111} = 1, t_{1112} = a, t_{1122} = b, t_{1222} = c, t_{2222} = 1. \tag{2}$$

By Theorem 1.3, the following lemma can be obtained easily.

Lemma 2.1. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor given by* (2). *Then $\mathcal{T}$ is copositive, i.e., $f(x_1, x_2) \geq 0$ for all $x = (x_1, x_2)^\top \geq 0$ if and only if*

(1) $\Delta' \leq 0$ *and* $a + c > 0$;

(2) $a \geq 0$, $c \geq 0$ *and* $1 + 3b \geq 0$;

(3) $\Delta' \geq 0$, $|a - c| \leq \sqrt{6b + 2}$ *and*

  (i) $-1 \leq 3b \leq 3$,

  (ii) $b > 1$ *and* $a + c \geq -\sqrt{6b - 2}$,

where $\Delta = 4 \times 12^3 \Delta'$, $\Delta' = (1 - 4ac + 3b^2)^3 - 27(b + 2abc - b^3 - c^2 - a^2)^2$.

Lemma 2.2. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $|t_{ijkl}| = 1$ and $t_{1111} = t_{2222} = 1$. Then $\mathcal{T}$ is copositive if and only if either*

$$b = t_{1122} = 1 \text{ or } a = t_{1112} = t_{1222} = c = 1.$$

Proof. It follows from Lemma 2.1 that $\mathcal{T}$ is copositive if and only if

(1) $\Delta' \leq 0$ and $a + c > 0$;

(2) $a \geq 0$, $c \geq 0$ and $1 + 3b \geq 0$;

(3) $\Delta' \geq 0$, $|a - c| \leq \sqrt{6b + 2}$ and $-1 \leq 3b \leq 3$;.

Since $|t_{ijkl}| = 1$, then the conditions (1)-(3) mean

(1) $\Delta' \leq 0$ and $a = c = 1 \Leftrightarrow a = c = 1$ and either

$$b = 1, \Delta' = (1 - 4 + 3)^3 - 27(1 + 2 - 1^3 - 1 - 1)^2 = 0,$$

or

$$b = -1, \Delta' = (1 - 4 + 3)^3 - 27(-1 - 2 + 1 - 1 - 1)^2 < 0;$$

(2) $a = c = 1$ and $b = 1$;

(3) $\Delta' \geq 0$, $|a-c| \leq \sqrt{6b+2}$ and $b = 1 \Leftrightarrow b = 1$ and either $ac = 1$,
$\Delta' = (1-4+3)^3 - 27(1+2-1-1-1)^2 = 0, |a-c| = 0 < \sqrt{6b+2} = \sqrt{8}$;
or $ac = -1$,
$\Delta' = (1+4+3)^3 - 27(1-2-1-1-1)^2 > 0, |a-c| = 2 < \sqrt{6b+2} = \sqrt{8}$.
So the conditions (1)-(3) are equivalent to

$$b = 1 \text{ or } a = c = 1.$$

This completes the proof. $\square$

Corollary 2.1. *Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $t_{1111} \geq 1$ and $t_{2222} \geq 1$. Then*

(1) $\mathcal{T}$ is strictly copositive if

$$t_{1112} \geq 1, t_{1222} \geq 1, t_{1122} \geq -1;$$

(2) $\mathcal{T}$ is copositive if

$$t_{1112} \geq -1, t_{1222} \geq -1, t_{1122} \geq 1.$$

Proof. Let $x = (x_1, x_2)^\top \geq 0$. Then

$$\mathcal{T}x^4 = t_{1111}x_1^4 + 4t_{1112}x_1^3x_2 + 6t_{1122}x_1^2x_2^2 + 4t_{1222}x_1x_2^3 + t_{2222}x_2^4.$$

(1) It is obvious that

$$\begin{aligned}
\mathcal{T}x^4 \geq & x_1^4 + 4x_1^3x_2 - 6x_1^2x_2^2 + 4x_1x_2^3 + x_2^4 \\
= & (x_1^2 + x_2^2)^2 + 4x_1x_2(x_1^2 - 2x_1x_2 + x_2^2) \\
= & (x_1^2 + x_2^2)^2 + 4x_1x_2(x_1 - x_2)^2.
\end{aligned}$$

So $\mathcal{T}x^4 > 0$ for $x \geq 0$ and $x \neq 0$. Suppose not, then there exists $x = (x_1, x_2)^\top \neq 0$ such that $\mathcal{T}x^4 = 0$, and hence,

$$0 = \mathcal{T}x^4 \geq (x_1^2 + x_2^2)^2 + 4x_1x_2(x_1 - x_2)^2 \geq 0.$$

That is,

$$(x_1^2 + x_2^2)^2 + 4x_1x_2(x_1 - x_2)^2 = 0,$$

which means $x_1^2 + x_2^2 = 0$, i.e., $x_1 = x_2 = 0$, a contradiction. Therefore, $\mathcal{T}$ is strictly copositive.

(2) For any $x \geq 0$, it follows from Lemma 2.2 that

$$\mathcal{T}x^4 \geq x_1^4 - 4x_1^3x_2 + 6x_1^2x_2^2 - 4x_1x_2^3 + x_2^4 \geq 0.$$

So, $\mathcal{T}$ is copositive. This completes the proof. $\square$

For a 4th-order 2-dimensional symmetric tensor $\mathcal{A} = (a_{ijkl})$ with its entires $a_{1111} > 0$ and $a_{2222} > 0$, let $t_{1111} = t_{2222} = 1$ and

$$t_{1112} = a_{1112}a_{1111}^{-\frac{3}{4}}a_{2222}^{-\frac{1}{4}}, t_{1122} = a_{1122}a_{1111}^{-\frac{1}{2}}a_{2222}^{-\frac{1}{2}}, t_{1222} = a_{1222}a_{1111}^{-\frac{1}{4}}a_{2222}^{-\frac{3}{4}}.$$

For $y = (y_1, y_2)^\top$ and $x = (x_1, x_2)^\top = (a_{1111}^{\frac{1}{4}}y_1, a_{2222}^{\frac{1}{4}}y_2)^\top$, then

$$\begin{aligned}\mathcal{A}y^4 =& a_{1111}y_1^4 + 4a_{1112}y_1^3y_2 + 6a_{1122}y_1^2y_2^2 + 4a_{1222}y_1y_2^3 + a_{2222}y_2^4\\ =& x_1^4 + 4t_{1112}x_1^3x_2 + 6t_{1122}x_1^2x_2^2 + 4t_{1222}x_1x_2^3 + x_2^4\\ =& \mathcal{T}x^4.\end{aligned}$$

Obviously, the copositivity of $\mathcal{A} = (a_{ijkl})$ coincides with one of $\mathcal{T} = (t_{ijkl})$, and hence, we can establish an analytically sufficient condition of the (strict) copositivity of a gerenal 4th-order 2-dimensional symmetric tensor $\mathcal{A} = (a_{ijkl})$, that can be very easily parsed and validated.

Corollary 2.2. *Let $\mathcal{A} = (a_{ijkl})$ be a 4th-order 2-dimensional symmetric tensor with its entires $a_{1111} > 0$ and $a_{2222} > 0$. Then*

(1) $\mathcal{A}$ is strictly copositive if

$$a_{1112} \geq a_{1111}^{\frac{3}{4}}a_{2222}^{\frac{1}{4}}, a_{1122} \geq -\sqrt{a_{1111}a_{2222}}, a_{1222} \geq a_{1111}^{\frac{1}{4}}a_{2222}^{\frac{3}{4}};$$

(2) $\mathcal{A}$ is copositive if

$$a_{1112} \geq -a_{1111}^{\frac{3}{4}}a_{2222}^{\frac{1}{4}}, a_{1122} \geq \sqrt{a_{1111}a_{2222}}, a_{1222} \geq -a_{1111}^{\frac{1}{4}}a_{2222}^{\frac{3}{4}}.$$

3 Copositivity of 4th order 3-dimensional symmetric tensors

3.1 Analytical expressions of strict copositivity

Theorem 3.1. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iiij} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. If there is at most one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$, then $\mathcal{T}$ is strictly copositive.

Proof. Without loss the generality, let $t_{1123} = -1, t_{1223} = t_{1233} = 1$. Then

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6t_{1122}x_1^2x_2^2 + 6t_{1133}x_1^2x_3^2 + 6t_{2233}x_2^2x_3^2\\ &+ 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3\\ &- 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2,\end{aligned}$$

and so,

$$\mathcal{T}x^3 = \frac{1}{4}\nabla\mathcal{T}x^4 = \begin{pmatrix} \sum\limits_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l \\ \sum\limits_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l \end{pmatrix}$$

It follows from Theorem 1.4 that we need only show that $\mathcal{T} = (t_{ijkl})$ is strictly semi-positive, i.e., for $x = (x_1, x_2, x_3)^\top \geq 0$, there exists $k \in \{1, 2, 3\}$ such that

$$x_k > 0 \text{ and } (\mathcal{T}x^3)_k > 0.$$

For each $i \in \{1, 2, 3\}$, we have

$$\begin{aligned}
(\mathcal{T}x^3)_1 = \sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3t_{1122}x_1x_2^2 + 3t_{1133}x_1x_3^2 \\
&\quad + 3x_1^2x_2 + 3x_1^2x_3 + 3x_2^2x_3 + 3x_2x_3^2 - 6x_1x_2x_3 \\
&\geq x_1^3 + x_2^3 + x_3^3 - 3x_1x_2^2 - 3x_1x_3^2 + 3x_1^2x_2 \\
&\quad + 3x_1^2x_3 + 3x_2^2x_3 + 3x_2x_3^2 - 6x_1x_2x_3 \\
&= (x_2 + x_3 - x_1)^3 + 2x_1^3 = x_1^3 + (x_2 + x_3 - x_1 + x_1) \\
&\quad \times ((x_2 + x_3 - x_1)^2 - x_1(x_2 + x_3 - x_1) + x_1^2) \\
&= x_1^3 + (x_2 + x_3)((x_2 + x_3 - x_1)^2 - x_1(x_2 + x_3 - x_1) + x_1^2);
\end{aligned}$$

$$\begin{aligned}
(\mathcal{T}x^3)_2 = \sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3t_{1122}x_1^2x_2 + 3t_{2233}x_2x_3^2 \\
&\quad + 3x_1x_2^2 + 3x_2^2x_3 - 3x_1^2x_3 + 3x_1x_3^2 + 6x_1x_2x_3 \\
&\geq x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_2 - 3x_2x_3^2 + 3x_1x_2^2 \\
&\quad + 3x_2^2x_3 - 3x_1^2x_3 + 3x_1x_3^2 + 6x_1x_2x_3 \\
&= (x_1 + x_3 - x_2)^3 + 2x_2^3 + 12x_1x_2x_3 - 6x_1^2x_3 \\
&= ((x_1 + x_3 - x_2)^3 + 2x_2^3) + 6x_1x_3(2x_2 - x_1);
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{T}x^3)_3 = \sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l &= x_1^3 + x_2^3 + x_3^3 + 3t_{1133}x_1^2x_3 + 3t_{2233}x_2^2x_3 \\
&\quad + 3x_1x_3^2 + 3x_2x_3^2 - 3x_1^2x_2 + 3x_1x_2^2 + 6x_1x_2x_3 \\
&\geq x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_3 - 3x_2^2x_3 + 3x_1x_3^2 \\
&\quad + 3x_2x_3^2 - 3x_1^2x_2 + 3x_1x_2^2 + 6x_1x_2x_3 \\
&= (x_1 + x_2 - x_3)^3 + 2x_3^3 + 12x_1x_2x_3 - 6x_1^2x_2 \\
&= ((x_1 + x_2 - x_3)^3 + 2x_3^3) + 6x_1x_2(2x_3 - x_1).
\end{aligned}
$$

So, it follows that

- $x_1 > 0$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_1 = 0, x_2 > 0$, $(\mathcal{T}x^3)_2 > 0$;
- $x_1 = 0, x_3 > 0$, $(\mathcal{T}x^3)_3 > 0$,

and hence, $\mathcal{T}$ is strictly copositive. $\square$

We definite

$$\mathcal{T}' = (t'_{ijkl}) \leq \mathcal{T} = (t_{ijkl}) \Leftrightarrow t'_{ijkl} \leq t_{ijkl}, \text{ for all } i, j, k, l.$$

Then for all $x \in \mathbb{R}^n_+$, we have

$$\mathcal{T}'x^4 = \sum_{i,j,k,l=1}^{n} t'_{ijkl}x_ix_jx_kx_l \leq \sum_{i,j,k,l=1}^{n} t_{ijkl}x_ix_jx_kx_l = \mathcal{T}x^4.$$

So, the (strict) copositivity of a tensor $\mathcal{T}'$ implies one of $\mathcal{T}$. Therefore, from Theorem 3.1, the following conclusions are obvious.

Corollary 3.2. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor. If $t_{iiii} \geq 1, t_{iiij} \geq 1, t_{iijj} \geq -1$ for all $i, j \in \{1, 2, 3\}, i \neq j$ and one of the following conditions,

(1) $t_{1123} \geq -1, t_{1223} \geq 1, t_{1233} \geq 1$;

(2) $t_{1123} \geq 1, t_{1223} \geq -1, t_{1233} \geq 1$;

(3) $t_{1123} \geq 1, t_{1223} \geq 1, t_{1233} \geq -1$,

then $\mathcal{T}$ is strictly copositive.

Theorem 3.3. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iiij} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{T}$ is srictly copositive if and only if

(1) $t_{1122} = t_{1133} = t_{2233} = 1$ if $t_{1123} = t_{1223} = t_{1233} = -1$;

(2) there is at least one 1 in $\{t_{1122}, t_{1133}, t_{2233}\}$ if two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are only -1.

Proof. Necessity. If $\mathcal{T}$ is srictly copositive, but the conditions don't hold. Then for $x = (1, 1, 1)^\top$, it follows that (1) one of $\{t_{1122}, t_{1133}, t_{2233}\}$ is -1,

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6t_{1122}x_1^2x_2^2 + 6t_{1133}x_1^2x_3^2 + 6t_{2233}x_2^2x_3^2 \\ &+ 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 \\ &+ 12t_{1123}x_1^2x_2x_3 + 12t_{1223}x_1x_2^2x_3 + 12t_{1233}x_1x_2x_3^2 \\ =& 27 + 12 - 6 - 36 = -3 < 0;\end{aligned}$$

(2) $t_{1122} = t_{1133} = t_{2233} = -1$,

$$\mathcal{T}x^4 = \sum_{i,j,k,l=1}^{3} t_{ijkl}x_ix_jx_kx_l = 27 - 18 - 24 + 12 = -3 < 0.$$

So, $\mathcal{T}$ is not srictly copositive.

Sufficiency. (1) Since $t_{1122} = t_{1133} = t_{2233} = 1$ and $t_{1123} = t_{1223} = t_{1233} = -1$, then

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 \\ &+ 4x_1^3x_2 + 4x_1^3x_3 + 4x_1x_2^3 + 4x_1x_3^3 + 4x_2^3x_3 + 4x_2x_3^3 \\ &- 12x_1^2x_2x_3 - 12x_1x_2^2x_3 - 12x_1x_2x_3^2,\end{aligned}$$

and so,

$$\begin{aligned}(\mathcal{T}x^3)_1 =& \sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l \\ =& x_1^3 + x_2^3 + x_3^3 + 3x_1x_2^2 + 3x_1x_3^2 + 3x_1^2x_2 + 3x_1^2x_3 \\ &- 3x_2^2x_3 - 3x_2x_3^2 - 6x_1x_2x_3 \\ =& (x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_1x_2^2 + 6x_1x_3^2 - 6x_2^2x_3 - 6x_2x_3^2 \\ =& (x_2 + x_3 - x_1)^3 + 2x_1^3 + 6x_2^2(x_1 - x_3) + 6x_3^2(x_1 - x_2);\end{aligned}$$

$$\begin{aligned}(\mathcal{T}x^3)_2 =& \sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l \\ =& x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_2x_3^2 + 3x_1x_2^2 + 3x_2^2x_3 \\ &- 3x_1^2x_3 - 6x_1x_2x_3 - 3x_1x_3^2 \\ =& (x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2x_2 + 6x_2x_3^2 - 6x_1^2x_3 - 6x_1x_3^2 \\ =& (x_1 + x_3 - x_2)^3 + 2x_2^3 + 6x_1^2(x_2 - x_3) + 6x_3^2(x_2 - x_1);\end{aligned}$$

$$
\begin{aligned}
(\mathcal{T}x^3)_3 = &\sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l\\
=&x_1^3+x_2^3+x_3^3+3x_1^2x_3+3x_2^2x_3+3x_1x_3^2+3x_2x_3^2\\
&-3x_1^2x_2-3x_1x_2^2-6x_1x_2x_3\\
=&(x_1+x_2-x_3)^3+2x_3^3+6x_1^2x_3+6x_2^2x_3-6x_1^2x_2-6x_1x_2^2\\
=&(x_1+x_2-x_3)^3+2x_3^3+6x_1^2(x_3-x_2)+6x_2^2(x_3-x_1).
\end{aligned}
$$

So, it follows that

- $x_1 \geq \max\{x_2, x_3\}$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_2 \geq \max\{x_1, x_3\}$, $(\mathcal{T}x^3)_2 > 0$;
- $x_3 \geq \max\{x_1, x_2\}$, $(\mathcal{T}x^3)_3 > 0$;

and hence, $\mathcal{T}$ is strictly copositive.

(2) We might take $t_{1122} = t_{1233} = 1$ and $t_{1133} = t_{2233} = t_{1123} = t_{1223} = -1$, then

$$
\begin{aligned}
\mathcal{T}x^4 =&x_1^4+x_2^4+x_3^4+6x_1^2x_2^2-6x_1^2x_3^2-6x_2^2x_3^2\\
&+4x_1^3x_2+4x_1^3x_3+4x_1x_2^3+4x_1x_3^3+4x_2^3x_3+4x_2x_3^3\\
&-12x_1^2x_2x_3-12x_1x_2^2x_3+12x_1x_2x_3^2,
\end{aligned}
$$

and so,

$$
\begin{aligned}
(\mathcal{T}x^3)_1 = &\sum_{j,k,l=1}^{3} t_{1jkl}x_jx_kx_l\\
=&x_1^3+x_2^3+x_3^3+3x_1x_2^2-3x_1x_3^2+3x_1^2x_2+3x_1^2x_3\\
&-3x_2^2x_3+3x_2x_3^2-6x_1x_2x_3\\
=&(x_2+x_3-x_1)^3+2x_1^3+6x_1x_2^2-6x_2^2x_3\\
=&(x_2+x_3-x_1)^3+2x_1^3+6x_2^2(x_1-x_3);
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}x^3)_2 = &\sum_{j,k,l=1}^{3} t_{2jkl}x_jx_kx_l\\
=&x_1^3+x_2^3+x_3^3+3x_1^2x_2-3x_2x_3^2+3x_1x_2^2+3x_2^2x_3\\
&-3x_1^2x_3-6x_1x_2x_3+3x_1x_3^2\\
=&(x_1+x_3-x_2)^3+2x_2^3+6x_1^2x_2-6x_1^2x_3\\
=&(x_1+x_3-x_2)^3+2x_2^3+6x_1^2(x_2-x_3);
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{T}x^3)_3 = & \sum_{j,k,l=1}^{3} t_{3jkl}x_jx_kx_l \\
= & x_1^3 + x_2^3 + x_3^3 - 3x_1^2x_3 - 3x_2^2x_3 + 3x_1x_3^2 + 3x_2x_3^2 \\
& - 3x_1^2x_2 - 3x_1x_2^2 + 6x_1x_2x_3 \\
= & (x_1 + x_2 - x_3)^3 + 2x_3^3 + 12x_1x_2x_3 - 6x_1^2x_2 - 6x_1x_2^2 \\
= & (x_1 + x_2 - x_3)^3 + 2x_3^3 + 6x_1x_2((x_3 - x_2) + (x_3 - x_1)).
\end{aligned}
$$

So, it follows that

- $x_1 \geq x_3$ and $x_1 > 0$, $(\mathcal{T}x^3)_1 > 0$, which is done; otherwise,
- $x_2 \geq x_3$ and $x_2 > 0$, $(\mathcal{T}x^3)_2 > 0$;
- $x_3 \geq \max\{x_1, x_2\}$, $(\mathcal{T}x^3)_3 > 0$;

and hence, $\mathcal{T}$ is strictly copositive. □

Combing the conclusions of Theorems 3.1 and 3.3, the main result is bulit in this subsection.

Theorem 3.4. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iiij} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{T}$ is strictly copositive if and only if

(1) there is at most one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;
(2) two of $\{t_{1123}, t_{1223}, t_{1233}\}$ are only -1 and there is at least one 1 in $\{t_{1122}, t_{1133}, t_{2233}\}$;
(3) $t_{1123} = t_{1223} = t_{1233} = -1$ and $t_{1122} = t_{1133} = t_{2233} = 1$.

3.2 Analytical expressions of copositivity

Theorem 3.5. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iijj} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. If there is at most one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$, then $\mathcal{T}$ is copositive.

Proof. Without loss the generality, let $t_{1123} = -1, t_{1223} = t_{1233} = 1$. Then

$$
\begin{aligned}
\mathcal{T}x^4 = & x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 + 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 \\
& + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 + 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3 \\
& - 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2,
\end{aligned}
$$

and so,

$$
\begin{aligned}
\mathcal{T}x^4 \geq & x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 \\
& - 4x_1^3x_2 - 4x_1^3x_3 - 4x_1x_2^3 - 4x_1x_3^3 - 4x_2^3x_3 - 4x_2x_3^3 \\
& - 12x_1^2x_2x_3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 \\
= & (x_1 + x_2 - x_3)^4 + 8(3x_1x_2^2x_3 - x_1^3x_2 - x_1x_2^3).
\end{aligned}
$$

Let $f(x_1,x_2,x_3)=(x_1+x_2-x_3)^4+8(3x_1x_2^2x_3-x_1^3x_2-x_1x_2^3)$. Then the stationary points of the function $f(x_1,x_2,x_3)$ are the solution to this system of equations,

$$\begin{cases} f'_{x_1}(x_1,x_2,x_3)=4(x_1+x_2-x_3)^3+8(3x_2^2x_3-3x_1^2x_2-x_2^3)=0,\\ f'_{x_2}(x_1,x_2,x_3)=4(x_1+x_2-x_3)^3+8(6x_1x_2x_3-x_1^3-3x_1x_2^2)=0,\\ f'_{x_3}(x_1,x_2,x_3)=-4(x_1+x_2-x_3)^3+24x_1x_2^2=0, \end{cases}$$

i.e.,

$$\begin{cases} 3x_1x_2^2+3x_2^2x_3-3x_1^2x_2-x_2^3=0,\\ 3x_1x_2^2+6x_1x_2x_3-x_1^3-3x_1x_2^2=0,\\ (x_1+x_2-x_3)^3=6x_1x_2^2, \end{cases}$$

and hence,

$$\begin{cases} x_2=0 \text{ or } 3x_1x_2+3x_2x_3-3x_1^2-x_2^2=0,\\ x_1=0 \text{ or } 6x_2x_3=x_1^2,\\ (x_1+x_2-x_3)^3=6x_1x_2^2. \end{cases}$$

So the stationary pointis $y=(0,0,0)$, and hence, $f(0,0,0)=0$. Since the boundary of the non-negative orthant $\{(x_1,x_2,x_3); x_1\geq 0, x_2\geq 0, x_3\geq 0\}$ are three coordinate planes, then at these boundary points, the function$f(x_1,x_2,x_3)$ give three 4th-order 2-dimensional symmetric tensors. It follows from Lemma 2.2 that

$$f(0,x_2,x_3)\geq 0, f(x_1,0,x_3)\geq 0, f(x_1,x_2,0)\geq 0.$$

Thus $\mathcal{T}x^4\geq f(x_1,x_2,x_3)\geq 0$ for any $x=(x_1,x_2,x_3)^\top\geq 0$. That is, $\mathcal{T}$ is copositive. □

From the proof of Theorem 3.5, the following cinclusion is easily obtained.

Corollary 3.6. Let $\mathcal{T}=(t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor. If for all $i,j\in\{1,2,3\}$ and $i\neq j$,

$$t_{iiii}\geq 1, t_{iijj}\geq 1, t_{iiij}\geq -1$$

and one of the following three conditions holds,

(a) $t_{1123}\geq -1, t_{1223}\geq 1, t_{1233}\geq 1$;

(b) $t_{1123}\geq 1, t_{1223}\geq -1, t_{1233}\geq 1$;

(c) $t_{1123}\geq 1, t_{1223}\geq 1, t_{1233}\geq -1$,

then $\mathcal{T}$ is copositive.

Theorem 3.7. Let $\mathcal{T}=(t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}|=t_{iiii}=t_{iijj}=1$ for all $i,j,k,l\in\{1,2,3\}$. Then $\mathcal{A}$ is copositive if and only if

(1) there is at least two 1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$ if there is only two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(2) there is at least five 1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$ if $t_{1123} = t_{1223} = t_{1233} = -1$.

Proof. Necessity. If $\mathcal{T}$ is copositive, but the conditions don't hold, then for $x = (1,1,1)^\top$,

(1) there are five -1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$,

$$\begin{aligned}\mathcal{T}x^4 =& x_1^4 + x_2^4 + x_3^4 + 6x_1^2x_2^2 + 6x_1^2x_3^2 + 6x_2^2x_3^2 \\ &+ 4t_{1112}x_1^3x_2 + 4t_{1113}x_1^3x_3 + 4t_{1222}x_1x_2^3 + 4t_{1333}x_1x_3^3 \\ &+ 4t_{2223}x_2^3x_3 + 4t_{2333}x_2x_3^3 \\ &+ 12t_{1123}x_1^2x_2x_3 + 12t_{1223}x_1x_2^2x_3 + 12t_{1233}x_1x_2x_3^2 \\ =& 21 + 4 - 20 - 24 + 12 = -7 < 0;\end{aligned}$$

(2) there are two -1 in $\{t_{iiij}; i,j = 1,2,3, i \neq j\}$,

$$\mathcal{T}x^4 = \sum_{i,j,k,l=1}^{3} t_{ijkl}x_ix_jx_kx_l = 21 + 16 - 8 - 36 = -7 < 0.$$

So, $\mathcal{T}$ is not copositive, which is a contradiction, and hence, the conditions hold.

Sufficiency. $\mathcal{T}x^4$ may be rewritten as follows,

$$\begin{aligned}\mathcal{T}x^4 =& (x_1 + x_2 - x_3)^4 + 4(t_{1112} - 1)x_1^3x_2 + 4(t_{1222} - 1)x_1x_2^3 \\ &+ 4(t_{1333} + 1)x_1x_3^3 + 4(t_{1113} + 1)x_1^3x_3 + 4(t_{2223} + 1)x_2^3x_3 \\ &+ 4(t_{2333} + 1)x_2x_3^3 + 12(t_{1123} + 1)x_1^2x_2x_3 \\ &+ 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} - 1)x_1x_2x_3^2;\end{aligned}$$

$$\begin{aligned}\mathcal{T}x^4 =& (x_1 - x_2 + x_3)^4 + 4(t_{1112} + 1)x_1^3x_2 + 4(t_{1222} + 1)x_1x_2^3 \\ &+ 4(t_{1333} - 1)x_1x_3^3 + 4(t_{1113} - 1)x_1^3x_3 + 4(t_{2223} + 1)x_2^3x_3 \\ &+ 4(t_{2333} + 1)x_2x_3^3 + 12(t_{1123} + 1)x_1^2x_2x_3 \\ &+ 12(t_{1223} - 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2;\end{aligned}$$

$$\begin{aligned}\mathcal{T}x^4 =& (x_1 - x_2 - x_3)^4 + 4(t_{1112} + 1)x_1^3x_2 + 4(t_{1222} + 1)x_1x_2^3 \\ &+ 4(t_{1333} + 1)x_1x_3^3 + 4(t_{1113} + 1)x_1^3x_3 + 4(t_{2223} - 1)x_2^3x_3 \\ &+ 4(t_{2333} - 1)x_2x_3^3 + 12(t_{1123} - 1)x_1^2x_2x_3 \\ &+ 12(t_{1223} + 1)x_1x_2^2x_3 + 12(t_{1233} + 1)x_1x_2x_3^2.\end{aligned}$$

Clearly, for the boundary points of the non-negative orthant, $x = (0, x_2, x_3)$, $(x_1, 0, x_3)$, $(x_1, x_2, 0)$, it follows from Lemma 2.2 that $\mathcal{T}x^4 \geq 0$.

(1) Without loss the generality, let $t_{1112} = t_{2223} = t_{1233} = 1$ and $t_{1113} = t_{1333} = t_{2333} = t_{1222} = t_{1123} = t_{1223} = -1$. Then

$$\mathcal{T}x^4 = (x_1 + x_2 - x_3)^4 + 8x_2^3(x_3 - x_1)$$

Let $f(x_1, x_2, x_3) = \mathcal{T}x^4$. Then the stationary points of the function $f(x_1, x_2, x_3)$ are the solution to this system of equations,

$$\begin{cases} f'_{x_1}(x_1, x_2, x_3) = 4(x_1 + x_2 - x_3)^3 - 8x_2^3 = 0, \\ f'_{x_2}(x_1, x_2, x_3) = 4(x_1 + x_2 - x_3)^3 + 24x_2^2(x_3 - x_1) = 0, \\ f'_{x_3}(x_1, x_2, x_3) = -4(x_1 + x_2 - x_3)^3 + 8x_2^3 = 0, \end{cases}$$

and hence,

$$\begin{cases} x_1 + x_2 - x_3 = \sqrt[3]{2}x_2 \Rightarrow x_1 - x_3 = (\sqrt[3]{2} - 1)x_2 \\ x_2^3 + 3x_2^2(x_3 - x_1) = 0 \Rightarrow x_2 = 0 \text{ or } x_1 - x_3 = \dfrac{1}{3}x_2, \end{cases}$$

i.e., $x_2 = 0, x_1 = x_3$. Therefore, the stationary points are $y = (x_1, 0, x_1)$ on the non-negative orthant, and hence,

$$f(x_1, 0, x_1) = 0 \Rightarrow \mathcal{T}y^4 = 0.$$

So $\mathcal{T}x^4 \geq 0$ at the stationary pointis and the boundary points, and then $\mathcal{T}x^4 \geq 0$ for all $x \geq 0$. That is, $\mathcal{T}$ is copositive.

(2) Let $t_{1113} = t_{1333} = t_{2333} = t_{1222} = t_{2223} = 1$ and $t_{1112} = -1$ without loss the generality. Then

$$\mathcal{T}x^4 = (x_1 - x_2 + x_3)^4 + 8(x_1x_2^3 + x_2^3x_3 + x_2x_3^3 - 3x_1x_2^2x_3).$$

Let $f(x_1, x_2, x_3) = \mathcal{T}x^4$. Similarly, solving the stationary point equations,

$$\begin{cases} f'_{x_1}(x_1, x_2, x_3) = & 4(x_1 - x_2 + x_3)^3 + 8x_2^2(x_2 - 3x_3) = 0, \\ f'_{x_2}(x_1, x_2, x_3) = & -4(x_1 - x_2 + x_3)^3 \\ & +8(3x_1x_2^2 + 3x_2^2x_3 + x_3^3 - 6x_1x_2x_3) = 0, \\ f'_{x_3}(x_1, x_2, x_3) = & 4(x_1 - x_2 + x_3)^3 + 8x_2(x_2^2 + 3x_3^2 - 3x_1x_2) = 0, \end{cases}$$

that is,

$$\begin{cases} (x_1 - x_2 + x_3)^3 + 2x_2^2(x_2 - 3x_3) = 0 \\ 3x_2(x_3^2 - x_1x_2 + x_2x_3) = 0 \Rightarrow x_2 = 0 \text{ or } x_1x_2 = x_3(x_3 + x_2) \\ x_2^3 + x_3^3 + 3x_1x_2(x_2 - 2x_3) = 0, \end{cases}$$

This yields $x_1 = x_2 = x_3 = 0$ on the non-negative orthant. Therefore, the stationary point is $y = (0, 0, 0)$, and hence, $\mathcal{T}y^4 = f(0, 0, 0) = 0$. So $\mathcal{T}x^4 \geq 0$ at the stationary pointis and the boundary points, and then $\mathcal{T}x^4 \geq 0$ for all $x \geq 0$. That is, $\mathcal{T}$ is copositive. $\square$

Combing the conclusions of Theorems 3.5 and 3.7, the main result is established in this subsection.

Theorem 3.8. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $|t_{ijkl}| = t_{iiii} = t_{iijj} = 1$ for all $i, j, k, l \in \{1, 2, 3\}$. Then $\mathcal{A}$ is copositive if and only if

(1) there is at most one -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$;

(2) there is only two -1 in $\{t_{1123}, t_{1223}, t_{1233}\}$ and there is at least two 1 in $\{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$;

(3) $t_{1123} = t_{1223} = t_{1233} = -1$ and there is at least five 1 in $\{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$.

3.3 Applications of a gerenal tensor

In this subsection, we apply Theorems 3.4 and 3.8 to find the (strict) copositivity of a gerenal 4th-order 3-dimensional symmetric tensor, and moreover, these analytic conditions can be very easily parsed and verified.

For a 4th-order 3-dimensional symmetric tensor $\mathcal{T} = (t_{ijkl})$ with its entires $t_{iiii} > 0$ for all $i \in \{1, 2, 3\}$, let $\mathcal{T}' = (t'_{ijkl})$ be a symmetric tensor with its entires $t'_{1111} = t'_{2222} = t'_{3333} = 1$ and

$$t'_{1112} = t_{1112}t_{1111}^{-\frac{3}{4}}t_{2222}^{-\frac{1}{4}}, t'_{1122} = t_{1122}t_{1111}^{-\frac{1}{2}}t_{2222}^{-\frac{1}{2}}, t'_{1222} = t_{1222}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{3}{4}},$$

$$t'_{1113} = t_{1113}t_{1111}^{-\frac{3}{4}}t_{3333}^{-\frac{1}{4}}, t'_{1133} = t_{1133}t_{1111}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{2}}, t'_{1333} = t_{1333}t_{1111}^{-\frac{1}{4}}t_{3333}^{-\frac{3}{4}},$$

$$t'_{2223} = t_{2223}t_{2222}^{-\frac{3}{4}}t_{3333}^{-\frac{1}{4}}, t'_{2233} = t_{2233}t_{2222}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{2}}, t'_{2333} = t_{2333}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{3}{4}},$$

$$t'_{1123} = t_{1123}t_{1111}^{-\frac{1}{2}}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{1}{4}}, t'_{1223} = t_{1223}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{1}{2}}t_{3333}^{-\frac{1}{4}},$$

$$t'_{1233} = t_{1233}t_{1111}^{-\frac{1}{4}}t_{2222}^{-\frac{1}{4}}t_{3333}^{-\frac{1}{2}}.$$

For $y = (y_1, y_2, y_3)^\top$ and $x = (x_1, x_2, x_3)^\top = (t_{1111}^{\frac{1}{4}}y_1, t_{2222}^{\frac{1}{4}}y_2, t_{3333}^{\frac{1}{4}}y_3)^\top$, then

$$\begin{aligned}
\mathcal{T}y^4 =& t_{1111}y_1^4 + 4t_{1112}y_1^3y_2 + 6t_{1122}y_1^2y_2^2 + 4t_{1222}y_1y_2^3 + t_{2222}y_2^4 \\
&+ 4t_{1113}y_1^3y_3 + 6t_{1133}y_1^2y_3^2 + 4t_{1333}y_1y_3^3 + t_{3333}y_3^4 \\
&+ 4t_{2223}y_2^3y_3 + 6t_{2233}y_2^2y_3^2 + 4t_{2333}y_2y_3^3 \\
&+ 12t_{1123}y_1^2y_2y_3 + 12t_{1223}y_1y_2^2y_3 + 12t_{1233}y_1y_2y_3^3 \\
=& x_1^4 + 4t'_{1112}x_1^3x_2 + 6t'_{1122}x_1^2x_2^2 + 4t'_{1222}x_1x_2^3 + x_2^4 \\
&+ 4t'_{1113}x_1^3x_3 + 6t'_{1133}x_1^2x_3^2 + 4t'_{1333}x_1x_3^3 + x_3^4 \\
&+ 4t'_{2223}x_2^3x_3 + 6t'_{2233}x_2^2x_3^2 + 4t'_{2333}x_2x_3^3 \\
&+ 12t'_{1123}x_1^2x_2x_3 + 12t'_{1223}x_1x_2^2x_3 + 12t'_{1233}x_1x_2x_3^2 \\
=& \mathcal{T}'x^4.
\end{aligned}$$

It is obvious that the copositivity of symmetric tensor $\mathcal{T} = (t_{ijkl})$ is equivalent to the copositivity of $\mathcal{T}' = (t'_{ijkl})$. So, applying Corollaries 3.2 and 3.6 (or Theorems 3.4(1) and 3.8(1)) to establish easily the following conclusions .

Theorem 3.9. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1, 2, 3\}$. Assume that one of the following three conditions holds,

(a) $t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$;

(b) $t_{1123} \geq t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$;

(c) $t_{1123} \geq t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$.

Then (1) $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iijj} \geq -\sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}};$$

(2) $\mathcal{T}$ is copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq -t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}}.$$

From Theorems 3.4 (3) and 3.8 (3), the following conclusions are established easily.

Theorem 3.10. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1, 2, 3\}$. Assume that

$$t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}.$$

Then (1) $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}, \quad t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}};$$

(2) $\mathcal{T}$ is copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}$$

and there is one $t_{kkkl} \in \{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$ such that

$$t_{kkkl} \geq -t_{kkkk}^{\frac{3}{4}} t_{llll}^{\frac{1}{4}} \text{ and } t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}} \text{ for all } t_{iiij} \neq t_{kkkl}.$$

From Theorems 3.4 (2) and 3.8 (2), the following conclusions are established easily.

Theorem 3.11. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 3-dimensional symmetric tensor with $t_{iiii} > 0$ for all $i \in \{1, 2, 3\}$. Assume that one of the following three conditions holds,

(a) $t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$;

(b) $t_{1123} \geq t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$;

(c) $t_{1123} \geq -t_{1111}^{\frac{1}{2}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{4}}, t_{1223} \geq t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{2}} t_{3333}^{\frac{1}{4}}, t_{1233} \geq -t_{1111}^{\frac{1}{4}} t_{2222}^{\frac{1}{4}} t_{3333}^{\frac{1}{2}}$.

Then (1) $\mathcal{T}$ is strictly copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iiij} \geq t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}};$$

and there is one $t_{kkll} \in \{t_{iijj}; i, j = 1, 2, 3, i \neq j\}$ such that

$$t_{kkll} \geq \sqrt{t_{kkkk} t_{llll}} \text{ and } t_{iijj} \geq -\sqrt{t_{iiii} t_{jjjj}} \text{ for all } t_{iijj} \neq t_{kkll}.$$

(2) $\mathcal{T}$ is copositive if for all $i, j \in \{1, 2, 3\}$ and $i \neq j$,

$$t_{iijj} \geq \sqrt{t_{iiii} t_{jjjj}}$$

and there is two $t_{kkkl}, t_{sssr} \in \{t_{iiij}; i, j = 1, 2, 3, i \neq j\}$ such that

$$t_{kkkl} \geq t_{kkkk}^{\frac{3}{4}} t_{llll}^{\frac{1}{4}},\ t_{sssr} \geq t_{sss}^{\frac{3}{4}} t_{rrrr}^{\frac{1}{4}}$$

and for all $t_{iiij} \in \{t_{iiij}; i, j = 1, 2, 3, i \neq j\} \setminus \{t_{kkkl}, t_{sssr}\}$,

$$t_{iiij} \geq -t_{iiii}^{\frac{3}{4}} t_{jjjj}^{\frac{1}{4}}.$$

4 Conclusions

For a 4th-order 3-dimensional symmetric tensor with its entries 1 or -1, the analytic necessary and sufficient conditions are established for strict copositivity and copositivity, respectively. These conditions can be applied to verify (strict) copositivity of a general 4th order 3-dimensional symmetric tensor.

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